The Dynamics of Social Exclusion

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Mathematical Annex

Consider a community that has two and only two schools of fixed size, both with N students. Let μN students leave each school each year, and let μN be recruited, where μ is a fixed fraction. Let the fraction of middle-class school-age children in the local population be θ, a fixed fraction. Let the fraction of middle-class students in year n be x_n in school 1, and y_n in school 2. Clearly, the number of middle-class students in school 1 in year $n+1$ is equal to the number in year n plus the number R_n recruited minus the number L_n leaving. Hence

$$Nx_{n+1} = Nx_n + R_n - L_n.$$

We must model R_n and L_n. Assuming the fraction of middle-class students among the school-leavers is the same as the fraction among the school population as a whole,

$$L_n = \mu N x_n.$$

Let us define r_n to be the fraction of recruits who are middle-class. Assuming this fraction is the same as the fraction among the local population as a whole, $r_n = \theta$, then

$$R_n = \mu N r_n = \mu N \theta.$$

If, however, there is a Matthew effect, where the fraction of middle-class students recruited to school 1 is higher if the fraction of middle-class students already in school 1 is higher than the fraction in school 2, this needs to be modified. In this case r_n will depend on x_n, $r_n = r_n(x_n)$. For brevity we shall drop the subscript n in the derivation of the model, and write simply $r = r(x)$.

We now model parental demand for places at different schools and the allocation of these places by the schools. This will allow us to explore the dynamics of this Matthew effect and the consequences for school recruitment.

Parental strategies

Schelling (1971, 1978), in a discussion of the development of racial (or other) segregation, makes use of "tolerance schedules", which give the fraction of whites (or blacks) who will tolerate living in a given neighbourhood as a function of the black:white ratio in that neighbourhood. We shall adapt this to consider "preference schedules", the fraction $m(x)$ of middle-class (or $l(x)$ of lower-class) parents who prefer to send their child to a school where the fraction of middle-class students is x (as opposed to sending them to the alternative school, where the fraction is $2\theta - x$).

We shall assume that the more middle-class students there are in a school, the more middle-class parents would prefer to send their child to that school. Hence the preference schedule m is an increasing function of x. It is clear that $m(x) + m(2\theta - x) = 1$, since each parent must want to send their child to one school or the other, and in particular $m(\theta) = \frac{1}{2}$. A simple function satisfying these conditions is $m(x) = \frac{1}{2} + a(x - \theta)$, with $a > 0$. It may have to be truncated in order to satisfy the condition that $m(x)$ is a fraction, $0 < m(x) < 1$ for any x, so to be more precise we define $m(x) = \text{mid}\{0, \frac{1}{2} + a(x - \theta), 1\}$.

(I have defined $\mathrm{mid}\{a, b, c\}$ to be the middle term of a, b and c, so that if e.g. $b < a < c$ then $\mathrm{mid}\{a, b, c\} = a$. Formally, $\mathrm{mid}\{a, b, c\} = \min\{\max\{a, b\}, c\}$.) But this truncation will make no difference to the result on stability to be derived below.

We shall also assume that the more middle-class students there are in a school, the more lower-class parents would prefer to send their child to that school, and use a similar preference schedule $l(x) = \mathrm{mid}\{0, \frac{1}{2} + b(x - \theta), 1\}$, with $b > 0$. But there are various reasons why the slope b of this preference schedule might be smaller than the slope a for middle-class parents. Although a preponderance of middle-class students tends to improve the quality of the school (which is the reason for the slope to be positive), there are counteracting effects such as fears that their child might feel excluded if they belong to a different social class from their peers, the high costs of moving to the catchment area if necessary, and maybe the lower priority given by the parents to the quality of the school. Hence we take $b < a$.

School strategies

Each year, there are $2\mu N\theta$ middle-class and $2\mu N(1 - \theta)$ lower-class parents with children to be recruited to one of the two schools. The school with a fraction x of middle-class students therefore receives $2\mu N\theta m(x)$ first-choice applications from middle-class parents, and $2\mu N(1 - \theta)l(x)$ first-choice applications from lower-class parents. (We have assumed that all parents make choices and apply accordingly. If more lower-class than middle-class parents fail to do so and are allocated according to catchment area this has the effect of amplifying the difference between a and b.) Schools may allocate places independently of social class, or by methods whose consequence is to maximise middle-class intake. We examine these in turn.

(a) Allocation independent of class

Let us first consider the case $x > \theta$, where school 1 is favoured. The school receives more applications than it has places available. Let us assume that it allocates places independent of class; then it gives a fraction $r(x)$ to middle-class students, $1 - r(x)$ to lower-class students, where

$$r(x) = \frac{\theta m(x)}{\theta m(x) + (1 - \theta)l(x)}. \tag{1}$$

It may be shown that $\max\{0, 2\theta - 1\} < r < \min\{2\theta, 1\}$, as required, so there is no necessity for any further truncation. Now consider the case where $x < \theta$, school 1 is less favoured. Then it only receives the middle-class students that are not taken by school 2,

$$r(x) = 2\theta - \frac{\theta m(y)}{\theta m(y) + (1 - \theta)l(y)} = 2\theta - \frac{\theta(1 - m(x))}{\theta(1 - m(x)) + (1 - \theta)(1 - l(x))}, \tag{2}$$

where $y = 2\theta - x$.

(b) Allocation giving preference to middle class

Let us now assume that each school takes all its middle-class applicants before filling up with lower-class students, and that neither has more middle-class applicants than it has places available. Since each school receives $2\mu N\theta m(x)$ middle-class applications for μN places, then the recruitment function is simply $r(x) = 2\theta m(x)$. This needs to be modified if either school receives more middle-class applicants than it has places, the excess going to the other school. Then

$$r(x) = \begin{cases} \min\{2\theta m(x), 1\} & \text{for } x > \theta, \\ \max\{2\theta - 1, 2\theta m(x)\} & \text{for } x < \theta. \end{cases} \tag{3}$$

Analysis

In the above equations, parental strategies depend crucially on the coefficients – the slopes – of the preference strategies. At one extreme, these may be so shallow that the parents are, in effect, expressing little or no preference for schools with a high proportion of middle-class pupils; at the other extreme, a school that wins an increased representation of middle-class pupils encounters a sharp increase in demand. School strategies have similarly been modelled according to their interest in, or blindness to, class background.

We now analyse the dynamic interactions of these various strategies, examining in particular the conditions under which the model moves away from the 'equity state' (similar class compositions for different schools) towards 'inequity states' (class polarisation of intakes).

(a) Allocation of school places independent of class

The model is nonlinear, and cannot be solved explicitly. However, all the information we require can be deduced from the general theory for equations of the form $x_{n+1} = f(x_n)$, where here

$$f(x) = x - \mu x + \mu r(x),$$

$r(x)$ is given by (1) for $x > \theta$ and (2) for $x < \theta$, and $m(x) = \text{mid}\{0, \frac{1}{2} + a(x - \theta), 1\}$, $l(x) = \text{mid}\{0, \frac{1}{2} + b(x - \theta), 1\}$. Steady states x^* of the model satisfy $x^* = f(x^*)$. It is easy to see that the equity state $x^* = \theta$ is a steady state, and we are interested in whether this is stable (when the schools tend to remain socially mixed) or unstable (when one school tends to monopolise middle-class students). Its stability is determined by $f'(\theta)$, and $f'(\theta) = 1 - \mu + \mu r'(\theta)$, where $r'(\theta) = 2\theta(1 - \theta)(a - b)$. Stability occurs if $f'(\theta) < 1$, $2(a - b)\theta(1 - \theta) < 1$; in this case any initial perturbation from equity tends to die away, with a half life given by $T_{\frac{1}{2}} = \log 2/\log(1/f'(\theta))$. Instability occurs if $f'(\theta) > 1$, $2(a - b)\theta(1 - \theta) > 1$; in this case any initial perturbation from equity tends to grow, becoming twice as large in a doubling time given by $T_2 = \log 2/\log(f'(\theta))$. Then there are two inequity steady states x_1^* and $x_2^* = 2\theta - x_1^*$ favouring school 1 and school 2 respectively, each of which is stable. The one that the system settles on is chosen by the Matthew effect, i.e. by whichever school obtains an initial small advantage. Inequity is more likely if the fraction θ of middle-class students in the population as a whole is close to 50%. For given θ, whether inequity develops is determined by the difference between the strengths a and b of the preference schedules for middle-class and lower-class students. The model does not predict that one school will monopolise all of the middle-class students. Rather, the size of the ultimate inequity is $z^* = x_1^* - x_2^*$, which is given by

$$z^* = \frac{2\theta(1 - \theta)(a - b) - 1}{\theta a + (1 - \theta)b},$$

again depending on the difference between a and b.

(b) Allocation of school places giving preference to middle class

Again there is a steady state, the equity state, at $x = \theta$, whose stability is determined by $f'(\theta)$, but now $f(x) = x - \mu x + \mu r(x)$ with r given by (3), so that $f'(\theta) = 1 - \mu + \mu r'(\theta) = 1 - \mu + 2a\theta\mu$. The solution is stable, and the solution of the system tends to it from any initial condition, as long as $f'(\theta) < 1$, $a\theta < \frac{1}{2}$, with half life $T_{\frac{1}{2}} = \log 2/\log(1/f'(\theta))$. The solution is unstable as long as $f'(\theta) > 1$, $a\theta > \frac{1}{2}$, with doubling time $T_2 = \log 2/\log(1/f'(\theta))$. In other words, the steady state is stable if the Matthew effect is weak. If the Matthew effect is strong, $a\theta > \frac{1}{2}$, there are two other steady states of the system. If $\theta < \frac{1}{2}$ these are at

$(x^*, y^*) = (0, 2\theta)$ and $(x^*, y^*) = (2\theta, 0)$, while if $\theta > \frac{1}{2}$ they are at $(x^*, y^*) = (2\theta - 1, 1)$ and $(x^*, y^*) = (1, 2\theta - 1)$; these are states of maximum class polarisation. The system will tend to one or other of these two steady states, chosen according to the Matthew effect. The model (3) for r may be interpreted as saying that each extra middle-class recruit to school 1 at time n leads to $2a\theta$ extra middle-class recruits at time $n + 1$. (The factor of 2 appears because each recruitment success for school 1 is also a recruitment failure for school 2.) Hence if $a\theta < \frac{1}{2}$ any inequality is reduced with time, whereas if $a\theta > \frac{1}{2}$ it grows, and we should expect the critical value of $a\theta$ to be $a\theta = \frac{1}{2}$. The strength of preference b shown by lower-class parents is now irrelevant: all that matters is obtaining middle-class applicants.

The stability condition $a\theta < \frac{1}{2}$ with policy (a) is more difficult to satisfy than the condition $(a - b)\theta(1 - \theta) < \frac{1}{2}$ with policy (b). As expected, policy (b) more often leads to class polarisation, and when it does so the polarisation is more extreme and the time scale over which it develops is shorter.

Other possible elaborations of the model

The model suggests that with strong parental preference for schools with a high proportion of middle-class pupils, and allocation policies that give preference to such middle-class pupils, the 'equity state' becomes unstable in face of even minor perturbations. Once any school gains an advantage in terms of its league table performance, movement towards an 'inequity state' follows, and this is then likely to be highly stable.

The model can be modified in a variety of ways. Thus, for example, recruitment to a school can be made dependent not on the proportion of its existing pupils who are middle class, relative to the proportion in other schools, but on the square of that relationship. The model can also examine what happens if the reputation of a school, which we have taken to determine the level of parental demand for places, is related not to measures of current school performance (which we have assumed to be simply and directly related to the proportion of middle-class pupils in the school) but to changes in that performance over recent years (is the school perceived to be 'improving' or 'on the slide'?). In neither case do these modifications change our general conclusions.